# GEOMETRIC SCALING AND ALLOMETRY IN LARGE CITIES

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## Abstract

Conventional ways of examining urban spatial structure in cities through locational patterns and density profiles have focused mainly on aggregate activity data such as the distribution of population and employment. With the appearance of large data bases dealing with building form and geometry, it is now possible to explore such patterns in terms of building size, impressing and extending the more physicalist tradition established in biology where scaling and allometry represent the signatures of growth and morphology, so that we can detect the underlying structure of cities as patterns of urban self-organization.

Scaling relationships in cities are important because they inform us about the way city size and location are constrained by geometry. Geometrical constraints also determine shape and these in turn relate to issues such as energy use, density of occupation, and circulation in buildings and streets. We start with a rather general and very simple approach to the density of population and buildings in cities which implies scaling, and then we develop two features of density, first focusing on the distributions of buildings by size relating directly to density, and second, on the relationships between different geometric characteristics of buildings based on the study of their allometry.

After we have sketched the key relationships, we examine the distribution of buildings in terms of their volume, height, and plot using rank-size relations. We also look at allometric relations between volume, height, and these same areas. We then selectively examine some of the rank-size and allometric relations for each of five different land uses and show that there are consistent differences in scaling which relate to the functions of these different land uses. One use of these relations is in estimating the distribution of energies by building size. There are also implications for the degree of heterogeneity and diversity characteristic of organically growing cities that should be maintained in planned cities. Finally we sketch how we might examine other geometric relations, in particular street systems, thus tying this work back to scaling relations which link physical to socio-economic attributes.

## Introduction

Although the predominant approach to cities is through an examination of their physical characteristics, it is still surprising that there has been so little systematic study of the effects of geometry in their form and function. The field of urban morphology is small and fairly descriptive despite the rather patchy development of more formalized theory such as that based on space syntax, fractal geometry, shape grammars and network science. Geometry constrains what is possible in cities in terms of their shape and size and the configuration and distribution of their internal elements, whether these be measured in terms of populations or buildings. It is important to have a comprehensive theory about how geometry influences urban distributions so that we can explore how shape changes with size for this is crucial to the planning of future forms and it is a crucial determinant of the way cities perform. Ideas about sustainability and the compact city which change the way we might move around cities are critically tied up with such questions of geometry.

In the development of urban theory, there are two quite different approaches to the question of geometry. First, it might be argued that the effect of space should be filtered out from our theory so that we can observe the effects of spatial and locational decision-making without the hindrance of geometric constraints. This approach assumes that such separation of form from function is possible, with form being relegated to the background and explanation in terms of function taking pride of place. Second, the much more dominant approach is based on the notion that space and geometry are intrinsic to theory development and that form cannot be disembodied from function. The first approach is not particularly common in comparison with the second although the great disadvantage of the second is that all the emphasis goes on form and little on function. For example, the focus turns entirely to spatial variation and spatial autocorrelation become key, with the consequent problem that non-spatial or a-spatial functions have little contribution to theory. In some instances, such explanation in purely spatial terms can be spurious.

In this paper, we are concerned to develop the second of these approaches but we will embrace our concern for the physical distribution of space in terms of building in large cities in the wider context of what we know about urban land use and their economic underpinnings. What we will do here is examine the distribution of the physical elements making up large cities which we will measure in the form of building blocks, first in terms of their size distributions. We need to know the form of their distribution which we suspect like many urban distribution functions is composed of many small elements and few big elements. We will measure these distributions using rank-size relations. We will do this for various geometric characteristics of the elements in question, namely dimensions such as length, area and volume that define their geometric form. Having examined these distributions, we will relate their geometric attributes to one another in the quest to measure their interdependencies on the assumption that as the size of the element in question increases, their geometric properties will scale within one another in different ways. This will engage us in developing their allometry.

The critical hypothesis relating to these geometric relations is that as the size of the typical elements changes, these relations may well depart from the standard geometric relations that characterize length, area, and volume. The allometric hypothesis suggests that there are critical ratios between geometric attributes that are fixed by the functioning of the element in questions and if the element changes in size, these ratios need to remain fixed for the element to still function. Often the geometry has to change if these ratios are fixed and D'Arcy Wentworth Thompson (1917, 1971) was amongst the first to demonstrate this. A good example relates to natural light penetrating buildings. As natural light depends on the surface area of a building, then to preserve a given ratio of natural light for the volume of the building, then the shape of the building has to change if the building is to be increased in size. In short, the surface area does not change at the same rate as volume and if the ratio has to be fixed to make the building function, then the volume has to change and this implies a different shape as the building increases in size.

We will not develop the detail of this argument much further here but our results are central to this notion. As yet there is no well worked out theory of urban allometry; indeed there is no complete theory of size in biological systems from whence these ideas arise (Bonner, 2006) although there are various theories in the making (West, Brown and Enquist, 1999). In fact many of the ideas in the emerging field of urban morphology support different aspects of this more general theory of size, shape and scale and our intention here is to simply put another piece in place in this much larger jigsaw (Kuhnert, Helbing, and West, 2006). To this end, having sketched the formal structure of scaling and allometry in urban geometry in the next two sections, we will first apply these ideas to a data base of building for Greater London, generating rank-size distributions for all buildings and their geometric properties and then examining the allometric relationships between these properties. We will then extend this to different land uses and conclude with some speculations on how street system geometries which tie buildings together in the city, also scale. This links our work to developments in space syntax that explore the scaling properties of street systems (Hillier, Turner, Yang and Tae-Park, 2007; Carvalho and Penn, 2004) Our ultimate quest is to use these various relations in examining a broad range of urban questions from energy outputs in buildings to the heterogeneity in land use and building types which has implications for sustainable communities and future city design.

Density, Size and Scale in Large Cities

The typical profile of population and other land use densities in a large industrial city is based on a relation between the density $\rho(d)$ and the radial distance $d$ from the central business district to any other place. Many different urban economic theories suggest that this density declines inversely with distance from the centre and this is borne out from wide empirical study beginning with Clark (1951). A typical function of density is based on an inverse power law

$$\rho(d) \sim d^{-\alpha} \tag{1}$$

where $\alpha$ is a parameter controlling the effect of distance. When $\alpha = 2$ , this is consistent with the inverse square law while the assumption that the relationship is a power law, implies scaling and connects this relation to those associated with space-filling which underpin fractal geometry (Batty and Longley, 1994). In fact power laws of this kind approximate exponential laws of the form $\rho(d) \sim \exp(-\lambda d)$ (and vice versa) and while these have proved more popular in urban economics as they emerge easily from various maximization frameworks, the assumption of a power law is more tenable for the idea of the fractal city (Batty, 2005).

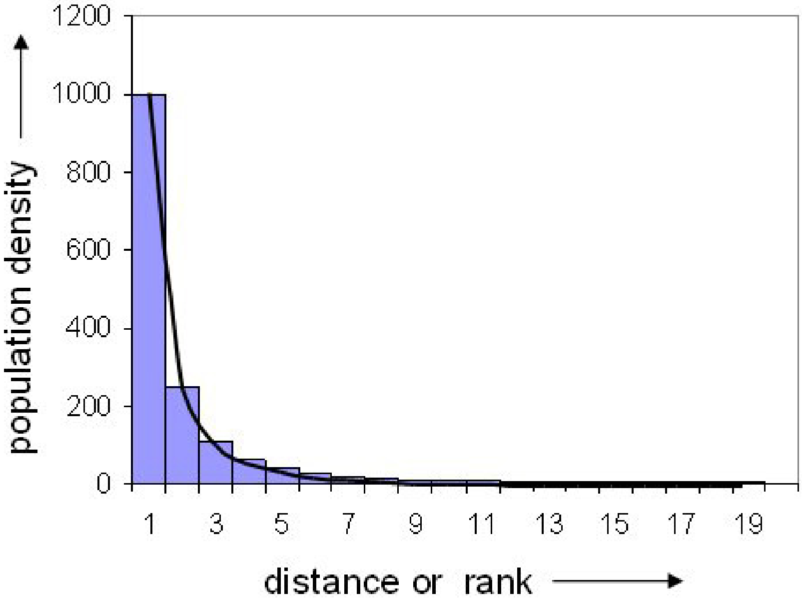
There is another way of examining this density relationship. First we assume that the city is one-dimensional, stretched out along a line from the CBD, and in such a case if we assume the density in question is associated with each unit distance band, then what we have is an ordered set of densities where the first distance band contains the largest density, the second the second largest and so on. If a distance band defines the unit object, then this set of units is naturally ordered from the largest to the smallest as distance from the

CBD increases. We show this for the function $\rho(d) = 1000\, d^{-2}$ in Figure 1 in continuous terms and as a histogram of objects for each unit distance band.

***Figure 1:***

*Population density in a one-dimensional city*






The *natural order* of the density objects from the CBD – let us assume they are single zones still – is equivalent to the *rank-order* of these objects because of the way the function has been applied. The rank-order is a size distribution and thus we see here that we have, unwittingly perhaps, constructed a rank-size distribution directly. Of course we are making the assumption that each distance band is a single object and the city is an idealized one-dimensional transect from the CBD to its periphery. If we spin the transect around its axis (at the CBD), we produce the familiar cone of densities with the number of objects from the CBD increasing as the square of the distance. However when we examine the number of objects in each distance band, then these simply increase at the same rate as across the transect. The rank-order, albeit with more objects in each band, still holds. This is equivalent to integrating the function in (1) with respect to its radial angle performing a sweep around each band or annulus. In practice, we would show this by discretizing each annulus into grid squares of equal size.

The best way to show this rank order is to first reset the notation in (1) from distance to rank – that is $\rho(r = d) = K\, r^{-\alpha}$ where $r$ is now the rank of the object in the set and K is a constant of proportionality, and then transform the function into the usual linear one using logarithms; that is

$$\log[\rho(r)] = \log K - \alpha \log r \qquad (2)$$

We need to say something about the parameter value $\alpha$ which we have assumed is 2. The pure rank-size rule assumes that this parameter is equal to 1 although the conventional wisdom tends to reflect the empirical evidence in city-size distributions where the parameter is usually greater than 1. In fact it is most unlikely that the density parameter in (1) would be 2 which was taken from the inverse square law and what evidence there is for fitting power laws to densities (see Batty and Longley, 1994) suggests that this value is between 1 and 2. There are implications here for space-filling in one-dimensional and two-dimensional space but these lie beyond our discussion here. The second issue involves fact that we have assumed that each zone is an object and that (2) is a distribution of zone sizes. As each unit distance band is the same in terms of size when it is discretely transformed as a set of grid cells, then population density $\rho(r)$ is the same as population $P(r)$. In this sense, we are dealing with population size when we carefully control to ensure equal zone size. Moreover, we might also make the assumption that the population $P(r)$ in each grid cell is contained in a number of buildings where each building is of the same size. Therefore the number of buildings in each cell would covary with the population and therefore we might expect the rank-size relation to scale the same way for buildings as for population.

However it is more likely that as population densities increase, then the number of buildings in terms of their size might also increase and that in an extreme case, we might have one building in each zone. What all this implies is that the relation between population (density) and the number of buildings is complex. It is likely that population would not be housed in identically size buildings which would destroy the rank-order in terms of buildings for all would be the same size; but that as population increases in density, buildings are likely to be larger in geometric terms, reflecting pressure on space and the tendency to occupy higher densities of space. In this sense, we might then expect that the parameter on the rank-size distributions of buildings would be lower than that which we have assumed for population. This of course is to be tested and is one of the central issues in this paper.

## Urban Geometry and Allometry

We have quickly moved from population densities to populations to buildings making the point that this translation is nowhere straightforward. We will talk this through verbally for the other set of relationships we intend to explore here relate to the geometric properties of buildings as they co-vary across the entire set. From Figure 1, let us consider the population density in different distance bands. If everyone occupied the same size of building, then the number of buildings would co-vary with the population, and there would be a uniform size of buildings everywhere. This would destroy the rank order but only at the building scale This is most unlikely because as populations agglomerate, then the pressure on space increases due to competition – populations wanting more space compete with each other, bidding up the rent they will pay for the location in question. This competition is essentially due to accessibility and to the need for people to agglomerate so that they can realize various externalities related to business, trade and exchange as well as linkages to efficient production. Thus it is likely that the buildings will increase in size as the population increases but not to the point where everyone occupies the same building. If we then assume that the number of buildings is related to the population size as $B(r) \sim P(r)^{\psi}$ where the parameter is usually $0 < \psi < 1$, then the average size of a building is

$$V(r) = \frac{P(r)}{B(r)} = P(r)^{1-\psi} \qquad (3)$$

In fact the size $V(r)$ scales with $r$, being proportional to $r^{-\alpha(1-\psi)}$. This is an allometric relationship in its own right but we will not explore it further in this paper. It is introduced simply to translate density of population to buildings.

With the parameter in (3) lying between 0 and 1, then it is quite clear that the parameter $\alpha(1-\psi)$ is less than $\alpha$ and thus the scaling of the geometry , in this case building size which we can assume is volume, is less than the population scaling. We can plot the population and building size rank-orders in logarithmic form (which is called a Zipf (1949) Plot after the scholar who popularized the rank-size rule) where we assume the rank is based on population in the first instance. In Figure 2, we show both these plots where it is clear than when $\alpha = 2$ and $\psi = 0.5$ , then the scaling for building size is $\alpha(1-\psi) = 1$.

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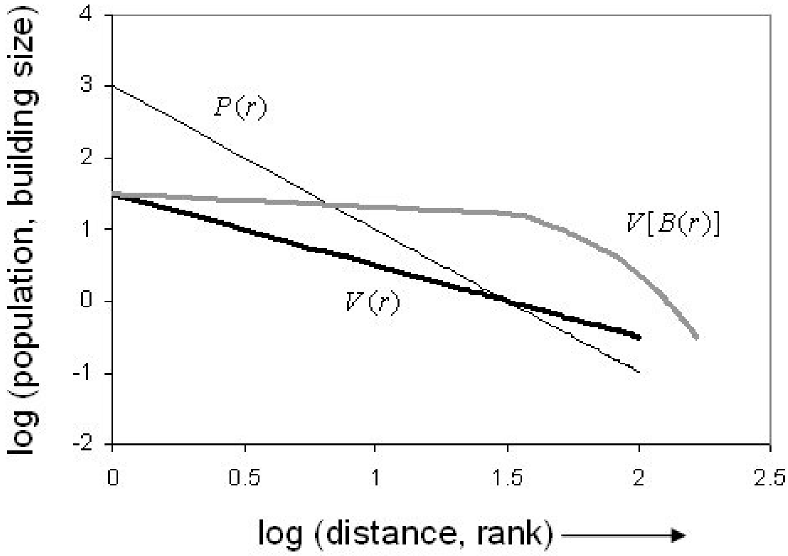



***Figure 2:***

*Log-log rank-size relations for population and, building size by distance rank and building size by numbers of building ranks*

In fact, examining building size by the rank that pertains to distance from which it is computed, is probably too simple in that with building size we have numbers that spread out the rank-orders. If we plot the building size against these new ranks where the numbers of buildings at each unit distance are accumulated, then we get the third plot in Figure 2 which looks much more like a lognormal with an average slope that is even smaller in magnitude than other two. There is not much point in taking this kind of theoretical analysis further as it is simply a matter of playing with functions but it does suggest that building size does scale in the same manner as other power law functions that determine the number of objects characterizing urban size distributions.

Some empirical evidence for this already exists and is widely available for the world's highest buildings and even a rank-size analysis has been done for the top 100 by Choi (2002) which we show in Figure 3(a) which is reminiscent of Figure 1 although most of these buildings are from different cities. To lighten the tone, alongside this, we place the top 20 skyscrapers in their original physical form as Figure 3(b).

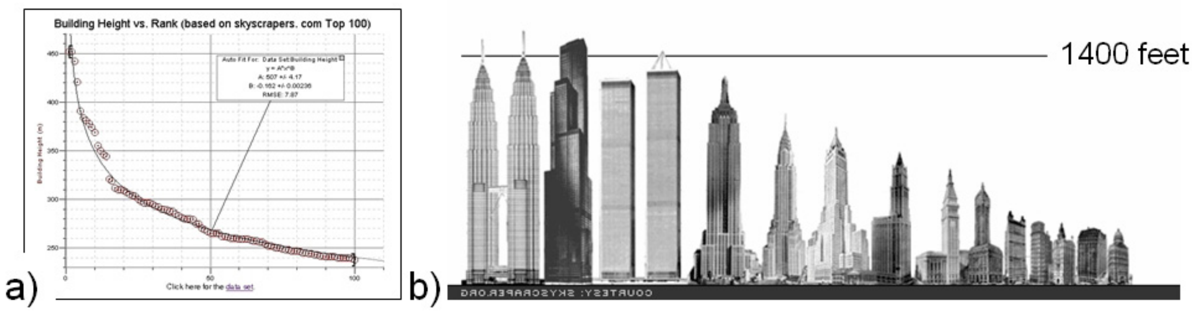



***Figure 3:***

*a) The top 100 buildings by height,*
*b) The top 14 in their physical form*
*(From http://i.cnn.net/cnn/interactive/world/0301/gallery.skyscrapers/intro.skyscrapers.new3.gif)*

In fact, a data base exists on the web for the top 200 buildings (skyscrapers) by height (see http://skyscrapers.com/en/bu/sk/st/tp/wo/) and it is easy to perform a log-log regression of the data as we show in Figure 4 below. This reveals remarkable consistency in terms of rank and size with a slope of $\alpha = \text{-}0.158$ and an adjusted correlation $r^2 = 0.996$. As we have argued theoretically, the slope is much smaller than 1 although we will return to this value a little later when we develop our own empirical work with the London data.

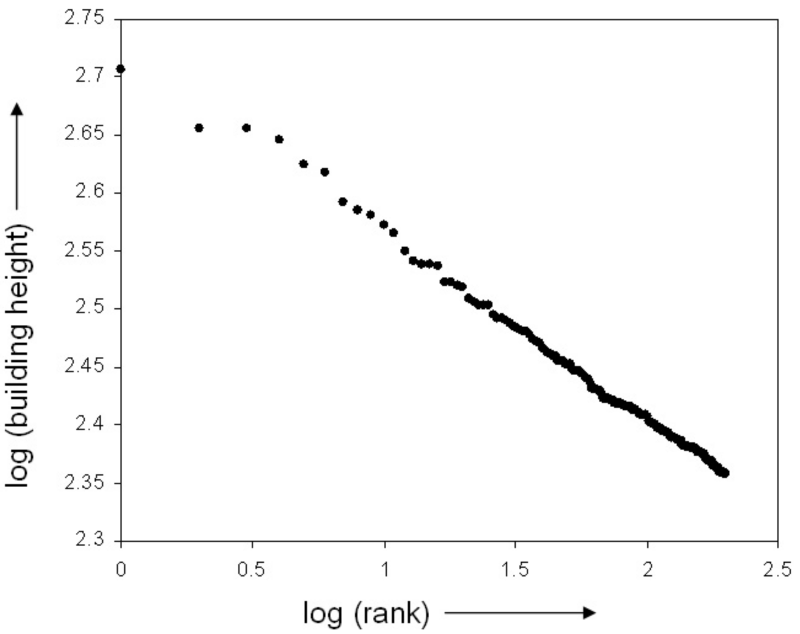



***Figure 4:***

*Zipf plot of building height versus rank for the World's top 200 skyscrapers*



What we are also interested in are the geometric features of overall building size, how they scale relative to their sizes and how they scale relative to one another. In terms of the key geometric features, first note that if was consider a building to be a square block, then we start with the standard geometric relations where the linear length $L$ of the building first determines the area $A$ as $L^2$ and then the volume $V$ as $L^3$ from which it is clear that $V = A\,L$. From these, standard allometric relations. first proposed by Huxley (1932, 1993), can be derived which imply what occurs if the volume, or area or length changes relative to each other of these measures. Then for our square block (which can be easily generalized to a less uniform geometry), $A = V^{2/3}$, $L = A^{1/2}$, and $L = V^{1/3}$. These imply that as the volume grows, the area grows at a rate which is 2/3rd's the rate of volume growth. This can easily be seen computing the relative growth rate or ratio of $dA/A$ to $dV/V$ (assuming a unit of time) which we can do as follows

$$\frac{dA}{dV} = \frac{2}{3}V^{(2/3)-1} = \frac{2}{3}\frac{V^{2/3}}{V} = \frac{2}{3}\frac{A}{V} \tag{4}$$

Rearranging the terms, we can get the ratio – the relative growth of $dA/A$ to $dV/V$ as

$$\frac{dA}{A}\bigg/\frac{dV}{V} = \frac{2}{3} \tag{5}$$

which can be easily generalized for any scaling parameter $\beta$. The general allometric relation relating some physical property $y$ of an object to another $x$ is thus

$$y = G\,x^{\beta} \tag{7}$$

where the scaling parameter is the relative growth rate of $y$ to $x$

$$\beta = \frac{dy}{y} \bigg/ \frac{dx}{x} \tag{8}$$



$\beta$ is also the elasticity as defined in economics. Equations (7) and (8) can thus be applied to any relationship which might be scaling with respect to different measures of size and where these sizes imply differential relative growth (Von Bertalanffy, 1973).

To simplify our treatment, we assume that the entire array of buildings can be represented as rectangular blocks. In fact this is the case as we will see in our buildings data base where buildings are constructed from plot area and modal height and where more complex buildings are glued together from simpler rectangular blocks. Then in terms of building blocks, linear dimension will involve heights (**z**) and vector lengths in the (*x, y*) plane from which area of the plot, the surface area of the block, and its volume or mass can be computed. We will not compute any internal measures of circulation in buildings for this does not exist in the databases as yet, nor will we compute any interior space as this does not exist either, although the data bases are being augmented to deal with such complexities in the future. We thus define for each building $B_j$ which is located at a point or centroid (toid) $j$ whose *x, y* coordinates we have, perimeter of the ground level plot $p_j$, average height of the building $h_j$ area of the plot $a_j$, surface area of the building $s_j$ (which is essentially the plot area at the bottom and at the top of the building and the areas of its faces), and volume or mass $v_j$. We are interested in their scaling with respect to rank-size which we have hypothesized above but we are also interested in how they scale with respect to each other. We will select from the following ten scaling relations but note that we will not present the results for surface areas as these have not yet been computed from the database. We will deal with these in a future version of the paper. Then

$$\left.\begin{array}{rrrr} p_j = Z_1\,h_j^{\kappa}; & p_j = Z_2\,a_j^{\eta}; & p_j = Z_3\,s_j^{\phi}; & p_j = Z_4\,v_j^{\mu} \\ & h_j = Z_5\,a_j^{\varphi}; & h_j = Z_6\,s_j^{\theta}; & h_j = Z_7\,v_j^{\chi} \\ & & a_j = Z_8\,s_j^{\xi}; & a_j = Z_9\,v_j^{\vartheta} \\ & & & s_j = Z_{10}\,v_j^{\zeta} \end{array}\right\} \tag{9}$$

where the $Z_*$ are the constants of proportionality and the power symbols are the appropriate allometric parameters – relative rates of change. We will not examine all these other possible relations either for all we wish to do at this stage is give some sense as to how these quantities vary.

Our key interest in urban allometry is to find out whether the scaling between areas and volumes implies changes in the shape of buildings. In terms of the relations in equations (9), we would expect the volume to scale as the cube of height and perimeter, and as the square of plot and surface areas. Surface area is likely to scale as the square of height and perimeter and linearly with plot area while the same relations pertain to plot area. Perimeter and height scale with each other linearly and these are the baseline allometries that we might expect. However if there are changes of shape, then these will be borne out by the parameters once we estimate the equations in (9) which we will do using linear regression of their logarithmic forms. In fact, it is likely that there will be considerable variation around these forms for all buildings. That is why we need to disaggregate the set of all buildings into different land use types which should reveal differences particularly between buildings in commercial and residential use.



We might also expect that surface area of buildings may scale quite differently from the 2/3rd's ratio that pertains to the standard pure allometric equations. This is because the skin of the building is the conduit for light and energy and buildings cannot explain their volume indefinitely through increasing the floor areas because such areas cannot be serviced through natural light and other forms of energy. Thus there are limits on shape in this regard. This is why it is likely that as buildings increase in size, they expand vertically rather than horizontally and these are the kind of deviations from standard allometry that we are seeking, although we are not yet able to show these results here. Our concern too is to count the number of buildings types by land use and to ultimately link these counts and their shapes to energy emission in buildings as well as issues involving circulation both within and between buildings.

## The Distribution of Geometric Properties in Buildings

### *The Buildings Data Base*

The data base we have assembled is based on our 3-D GIS/CAD model of London, which we refer to as Virtual London (Batty and Hudson-Smith, 2005). This is a digital model of all building blocks within about 30 kilometers of the CBD – the City of London or 'square mile' – which cover the 33 boroughs comprising the Greater London Authority (GLA) area. The data set is unique in that it has been created automatically from two main sources of data: first the vector parcel files that are part of *MasterMap* (http://www.ordnancesurvey.co.uk/oswebsite/products/osmastermap/) from Ordnance Survey which codes all land parcels and streets to at least one meter accuracy; and second a data set of buildings heights constructed from InfoTerra's LIDAR data which produces a massive cloud of 3-D *x-y-z* data points which when used in association with the vector parcel data, can be used to extrude all buildings. In this data set, there are some 3,601,389 distinct buildings centroids (or toids as they are called). We are currently dealing with all 3.6 million but in future work, we will be aggregating toids to ensure that we are dealing with appropriate blocks and this becomes critical when land use is to be assigned because land use is tagged to street addresses which is a subset of all toids.

To give some idea of the range of this data set, the maximum height of any block is 204.06 meters and this is probably the Canary Wharf Tower in the London Docklands. The mean height is 5.76 meters and the standard deviation is 3.29 meters which shows that the frequency of building heights is very skewed to the left, reflecting the fact that this distribution is likely to follow a power law. For illustrative purposes only the top 10 blocks by height in London are 204, 197, 169, 160, 151, 150, 138, 130, 128, and 123 meters in comparison with the top 10 from the skyscrapers database which are 509, 452, 452, 442, 421, 415, 391, 384, and 381. London's highest building is in fact not in the top 200 in the world and from the regression in Figure 4, we can estimate its rank as 400. London is not a city of tall buildings!

From the data set, we are currently working with the perimeter of each plot which is computed directly from the *MasterMap* data, and the mean height of a plot which is important as there are many different heights from the LIDAR data reflecting complex roof shapes, masts, air conditioning units and so on. Other measures of height such as median and mode do not change the results below substantially. We compute volume simply from taking the area of the plot and multiplying it by its height. This does not take account of course of the fact that some buildings will taper but currently we are not able to do much about this as we do not have elaborate algorithms in place to construct intricate roofing shapes. We are able to compute these measures – perimeter $\{p_j\}$, height $\{h_j\}$ and volume $\{v_j\}$ which we will use for the rank-size and allometric analysis which we deal with in the following section.

When it comes to land use, we need to be clear about how we tag a land use to each building. From the *MasterMap* Layer 2, we have land uses associated with each street address for which there is a toid. However there are many blocks that do not have street addresses and these tend to be part of other building complexes and/or are very small and somewhat idiosyncratic in their form, such as sheds, lean-to's and such-like bric-a-brac. Currently we are at work on an algorithm to clean up this data and to produce a much tidier set of building blocks and this work will be reported in future papers. What we have done here is to classify the land use into nine different land uses which we list in terms of toids classified with at least one *residential*, *office*, *retail*, *services*, *industrial*, *educational*, *hotel*, *transport*, and *general-commercial* land use. We have not yet broached the difficult question of multiple land uses, for we have simply taken these combined classes. In future work, we will develop a much more sensitive classification which takes account of multi-use and attempts to make sense of such ambiguities. We think in fact our results are robust in any case as the majority of land uses have single uses associated with their toids.

### *The Distribution of Building Properties: Rank-Size Relations*

We start by defining average or mean height, perimeter of the plot, and area of the plot, and we then simply determine the volume of each building by multiplying area by height. This of course is an approximation but until we get better algorithms for interpolating roof shapes it needs to suffice. We have computed the ranks of perimeter, height, area and volume and we illustrate these for all 3.6m blocks in Figure 5 where we note that in this figure, the scaling parameter is given as $1/\alpha$. We generate the following relations: $p_j(r) = K_1 r^{-0.432}$, $h_j(r) = K_2 r^{-0.223}$, $a_j(r) = K_3 r^{-0.684}$, $v_j(r) = K_4 r^{-0.799}$ where the parameters values are less than unity but not inconsistent in terms of their relative magnitude. In essence we might expect volume to decline more steeply with rank than area, which in turn is likely to fall more steeply than height or perimeter for this is the sequence of objects from 3 to 2 to 1 dimension. Assuming that $v_j(r) \sim p_j^3$, and $a_j(r) \sim p_j^2$, then if we plug these idealized perimeters into the expressions for area and volume in the estimated rank-size relations, then we do appear to get similar orders to the estimated parameters values, that is the values are consistent with one another in terms of the assumed geometry.

Figures 5(a) to (d) are quit self-explanatory. We have very dramatic linearity in the log-log plots over several orders of magnitude for volume from $10^7$ to $10^2$ after which the plot falls very steeply, implying that buildings less than 100 sq meters in volume behave quite differently. These are really sheds and bric-a-brac referred to earlier and in future work will be discounted to an extent as we construct better building blocks (Steadman et al., 2000). In fact we show a sample of this kind of detail in Figure 6 where the thick red lined blocks are in fact not addressed as land uses in the data and can be assumed to be part of other buildings or free standing sheds which should not be considered in the same manner as the rest of the buildings.

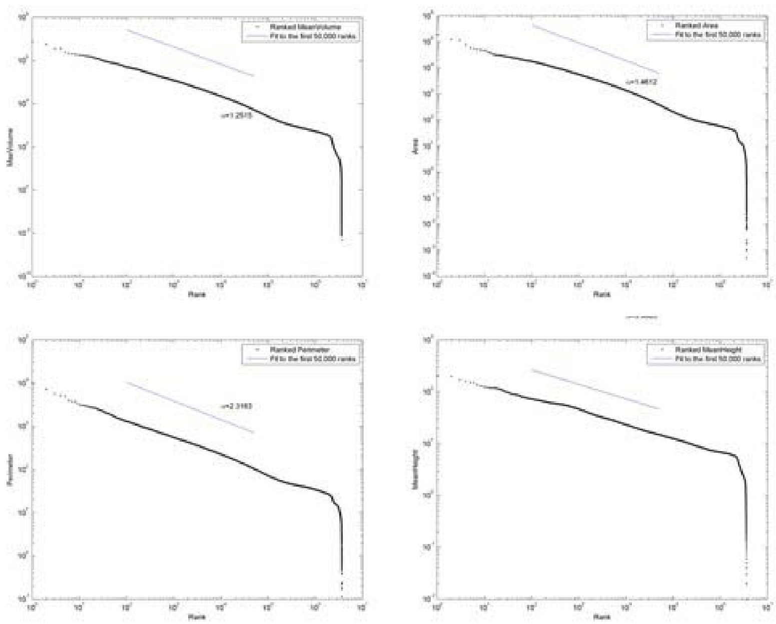


***Figure 5:***

*Rank size relations of volume, area, perimeter and height*

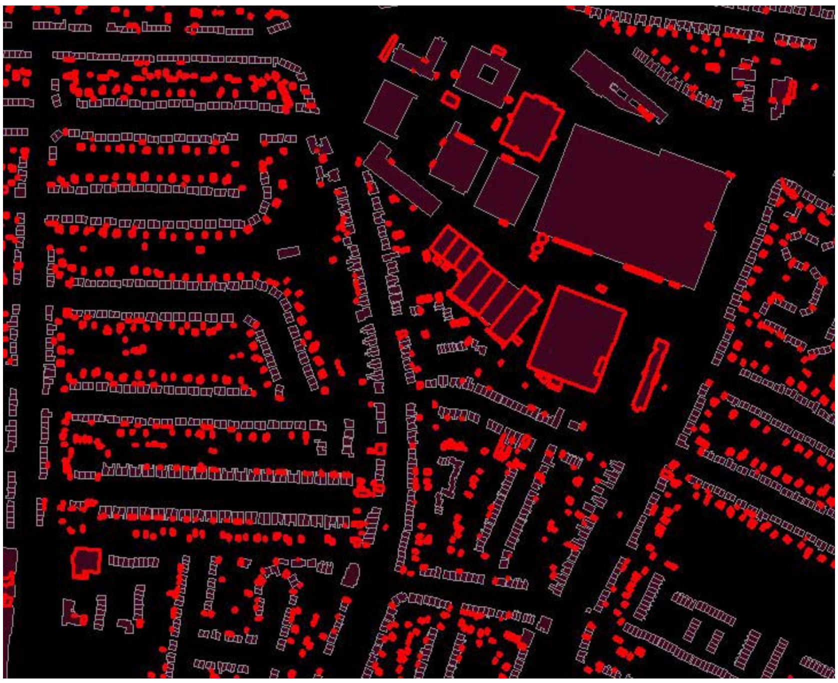


***Figure 6:***

*Buildings and freestanding structures of small size not 'addressed' by toid in the land use data base*

The same steep fall in slope for volume at the highest ranks occurs for the other plots as well. For area, the rank-size is linear from $10^5$ to $10^2$, for perimeter from $10^4$ to $10^1$, and for height from $10^2$ to $10^1$ which means that in general the building heights in this data set are less than 100 meters. As we know the tallest building in London at Canary Wharf is just over 200 meters tall. These regressions are striking in their linearity and such rank-size relations are amongst the best we have come across. In fact this bears out the remarkable linearity of the rank-size of the heights of the top 200 buildings in the world which enabled us to make such good predictions of building heights further down the scale. In away all this is preamble to the central issues of allometry that we will now examine first for the aggregate building set.

## The Allometry of Buildings



### *Relations between Perimeter, Area and Volume*

We will not estimate the relation between height and area or volume but instead use perimeter; volume is constructed from area and height and height is thus redundant as an independent measure although in future work it will still be considered when we refine the data set. We can restate our allometric relations between area and perimeter, volume and perimeter, and then volume and area as follows, and by the side of these we show the estimated relations from the data set

$$\left.\begin{array}{ll} a_j \sim p_j^2 \quad ; & a_j \sim p_j^{1.833} \\ v_j \sim p_j^3 & v_j \sim p_j^{2.386} \\ v_j \sim a_j^{3/2} & v_j \sim a_j^{1.296} \end{array}\right\} \qquad (10)$$

It is immediately clear that the order of the parameters matches the order of the geometric scaling. That is, the parameter of area on perimeter is less the square while the value of the relation between volume and area is less than 3/2. This means that as the perimeter increases, the area increases less than the normal geometric relation implying that shape should change and probably become more crennelated – implying a longer perimeter – as the area grows. In terms of volume, this increases at less than 3/2 of the area which suggests that the volume must get proportionately less as the area grows. This bears out the implied observation that as the surface grows the shape must change.

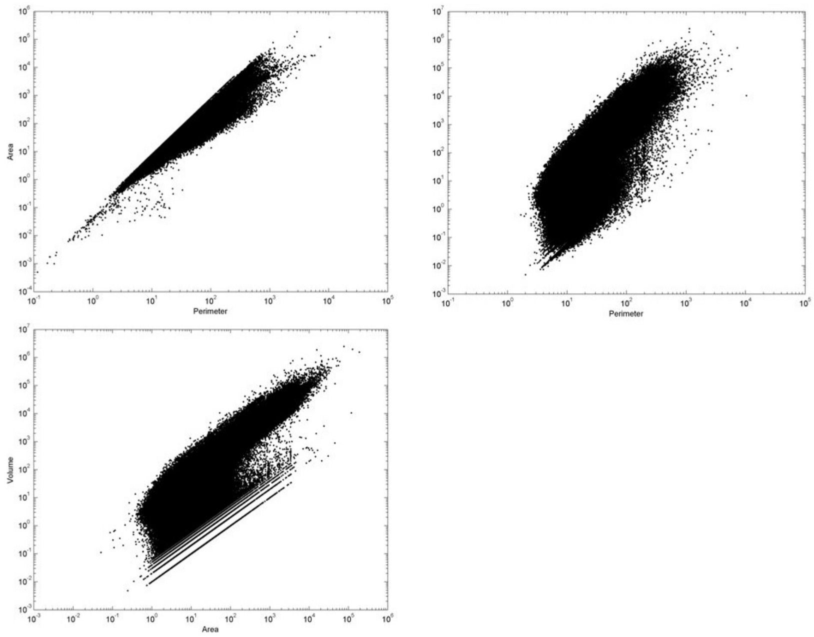



***Figure 7:***

*The key allometric relations*

We show these relations in Figure 7 where it is immediately clear the trend is rather firm but the scatter is also substantial. We have not computed the correlations – we will do so in later work – but it is clear that these relations are strong. A word of explanation is in order, These graphs contain 3.6m points and are very dense clusters that would best be described by contour surfaces showing their density and this we will do in future work. We have also systematically looked at height versus perimeter, area and volume, and these correlations together with those in Figure 7, albeit at a smaller scale are shown in Figure 8. This shows that the relationships between height and the other variables are much more scattered and do not show much trend. This is to be expected as height is the variable in the London database that is the most variable and does not seem to relate very strongly to the standard allometric relations. This requires considerable further research as it is central to some of the notions in this paper which relate to how volume scales with plot area and to questions of surface area of buildings that define their skin which we are going to develop in the next stage.



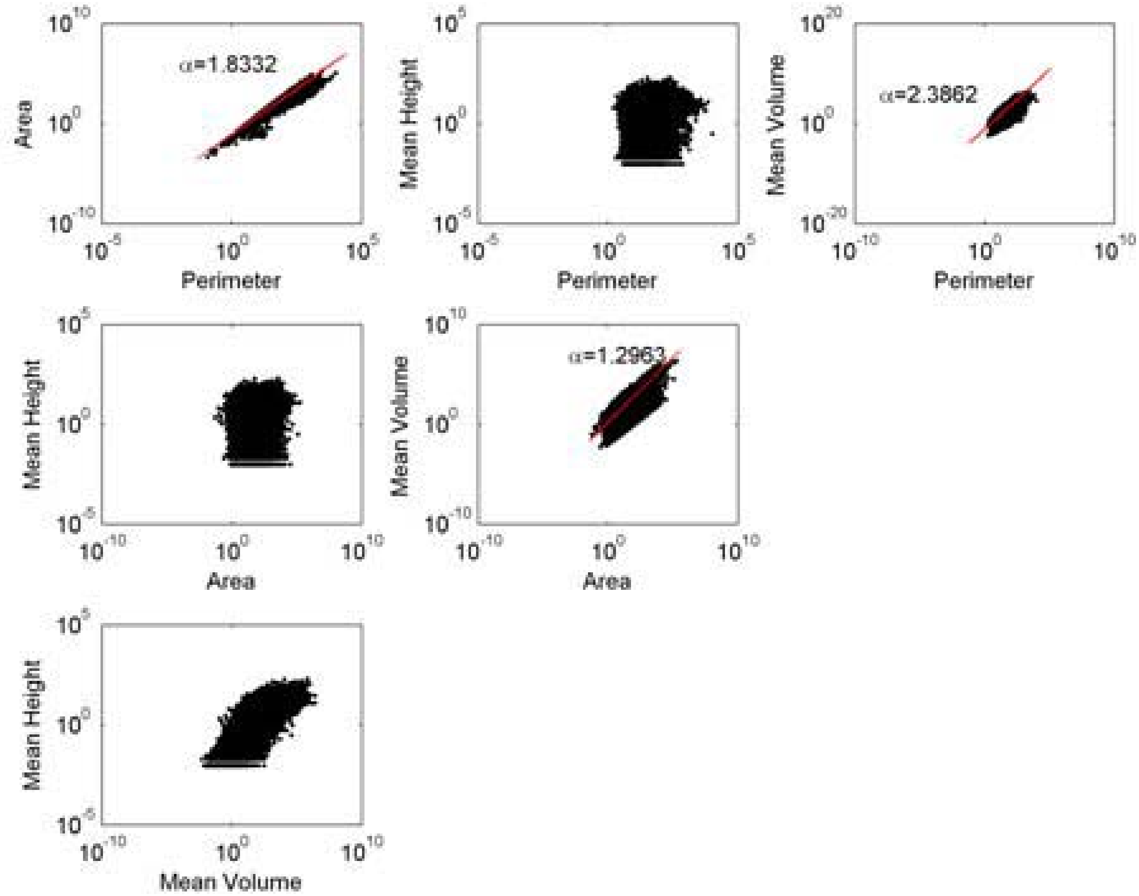



***Figure 8:***

*Exploring the complete set of relations*

### *Disaggregating to Land Uses*

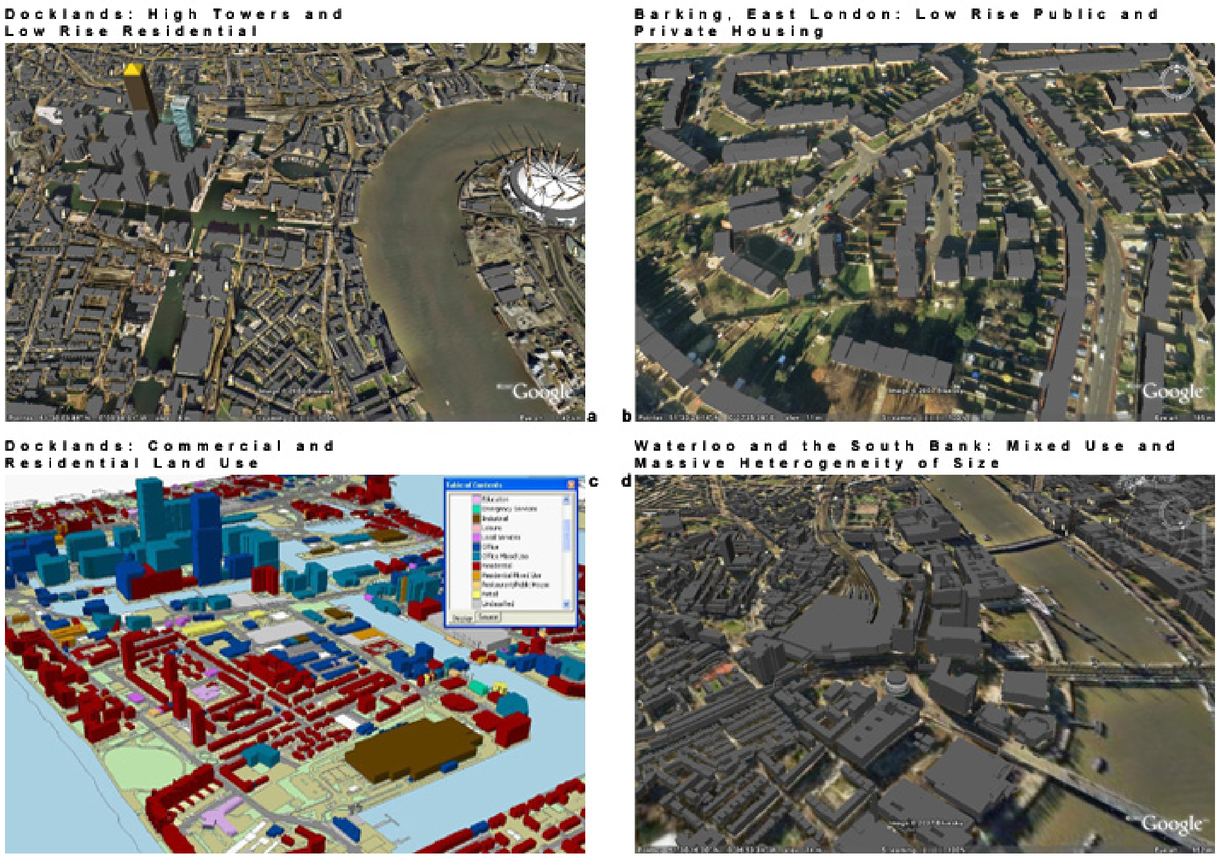
The next step is to tag the buildings data with land use and this is where some of the problems identified earlier with freestanding small blocks are at their worst. Land use is available in *MasterMap* Layer 2 with some hundreds of categories but these are tagged to street address. Each street address is a toid which marks a plot or building but all toids do not have street addresses. In Figure 6, the blocks in red do not have a street address and thus these are excluded from our first analysis of the land use data. Moreover the land use data varies considerably across the city from areas which are very mixed to those which are uniform. To give the reader some sense of the variety in this data, in Figure 9 we show four pictures of such use. In Figure

9(a), we show the London Docklands at Canary Wharf which is London's second CBD but also contains considerable low rise housing. There is massive variation in building size and geometry here in contrast to Figure 9(b) which is Barking in East London which is mainly residential and quite uniform at two storeys. In Figure 9(c), we show the land uses in Canary Wharf aggregated to similar categories to those we are using here and this shows the variety again but also shows that commercial tend to be in larger, higher blocks than residential. Figure 9(d) is an area around the South Bank and Waterloo which has very different shaped buildings characteristic of transport and office uses mixed with some residential and retail.

All we have done so far is to explore the distribution of buildings tagged to our nine land use categories – residential, office, retail, services, industrial, educational, hotel, transport, and general-commercial in terms of their rank-size of height and area. We have not computed volume from area times (x) height in this data set as yet. What we find is that there is very little difference between each land use. We show the Zipf Plots of these two sets of relations for the nine land uses in Figure 10. Because the range of these data vary considerably, we have in fact collapsed them onto the same scale for each land use by plotting the ratio of the size to its mean against the ratio of the rank to its mean and this enables us to see how close each land use distribution is to any of the others. What we find is that height scales as $h_j(r) = K_2 r^{-0.243}$, and $a_j(r) = K_3 r^{-0.593}$ for all land uses – or for the average of all the merged data, and these values are quite close to the overall scaling for all buildings. The distributions again show the steep fall at a similar point in the distributions and strong linearity over the rest of the plots. Again there is clear evidence of scaling over several orders of magnitude. There is much work to do on these data and the next step is to explore their allometry.

***Figure 9:***

*Variation in buildings sizes and land use visualized from the database*



### *Scaling and Street Systems*

We will conclude this first foray into the geometry of buildings in large cities with a move from volumes and areas to linear features in the form of street systems. Carvalho and Penn (2004) found strong scaling in axial lines from various databases produced using space syntax analysis but left the key question remaining as to whether street segment lengths actually scale for definitions based on more mundane considerations defined by street intersections. Axial lines are lines of sight or uninterrupted movement and are not the same as those which are defined by building street networks from the bottom up, Ordnance Survey for example in their ITN (transport) layer of *MasterMap* build streets from center lines which are usually defined between street intersections or at least points where streets converge on one another in some way. In the London data, there are 64753 street segments that exclude alleys and lanes which range from about 1 meter in length up to 3773 meters at maximum. When we examine the size distribution of these street segment length, we find that there is scaling of a sort but that it is nearer to log normality in the distribution. We show the rank-size relation in Figure 11 for the first 32000 segments.



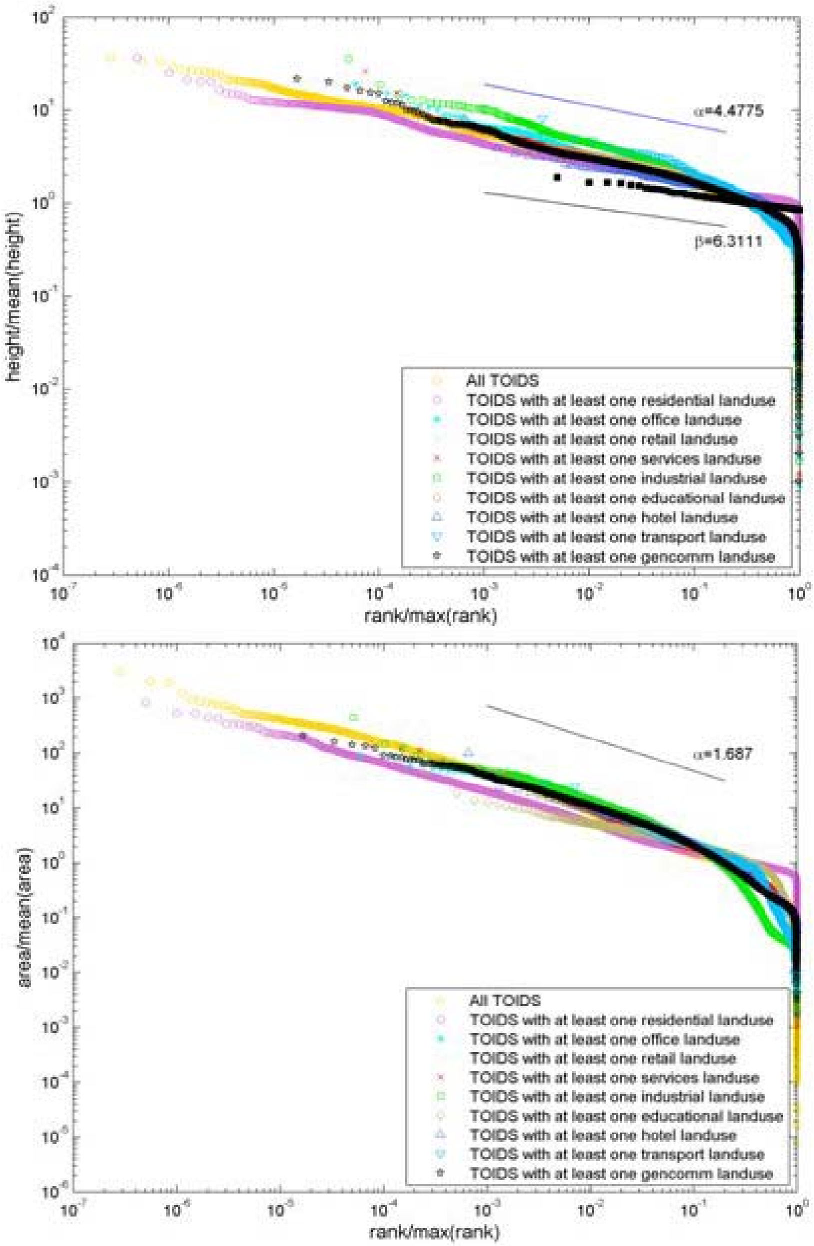
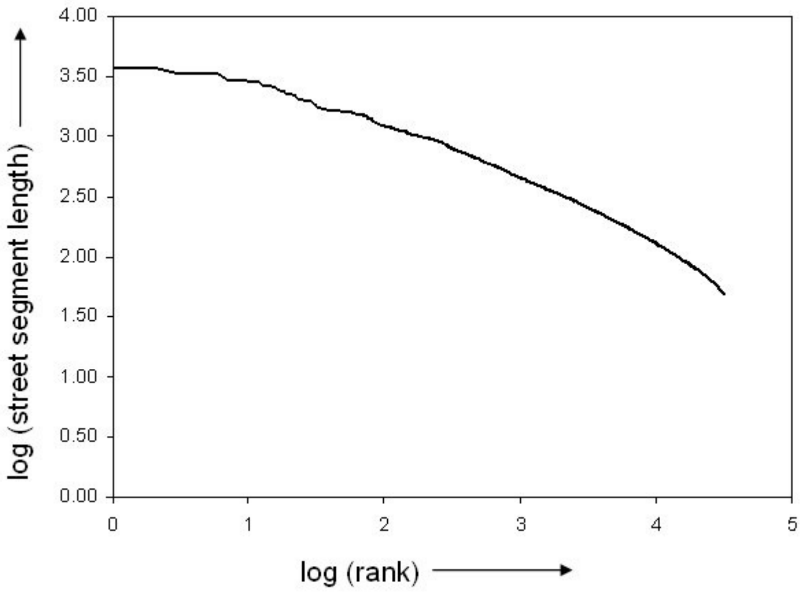



***Figure 10:***

*Rank size distributions for nine land use categories collapsed onto a common scale*

***Figure 11:***

*Rank size distribution of street segment lengths in greater London*




What we have not yet done is demonstrate how scaling in buildings relates to scaling in street systems and to do this, we must establish a much clearer link between how the geometry of buildings generates street systems and vice versa. This should be possible to some extent but the geometry of streets is much more arbitrary than the geometry of buildings. Where a street begins and ends is always a problem which is faced in space syntax, for example, and although there is more common agreement about street segments being defined by intersections, this too is rather an arbitrary feature of the analysis. There is thus an urgent need for buildings, land uses and street systems to be considered together and this would also involve non-built up areas of open space.

## Conclusions and Next Steps

We are clearly in the midst of researching this area. Besides developing the analysis of existing data sets in a much fuller manner, exploring the allometry of buildings associated with different land uses, for example, we need to revisit the data base in considerable detail and iron out many of the problems of building size and type that we have identified here. We also need to extend the analysis to deal with different rank-size and allometric relations in different parts of the city, showing how these relations will change for different kinds of areas, for example as we implied in the pictures of different parts of London in Figure 9. In short we need to ground the analysis in terms of its spatial and geographic context.

We are much encouraged by the very strong scaling implicit in this data and of course to confirm this we need more example from other cities. We need to relate the physical geometry to other measures, particularly linear measures such as utilities and street systems as well as socio-economic activity volumes as proposed by Kuhnert, Helbing, and West (2006) and Bettencourt, Lobo, Helbing, Kuhnert, and West (2007). We need to link the analysis much more strongly to fractal geometry and we need to link it to circulation patterns in buildings (Bon, 1973; Steadman, 2006). In particular the most urgent extension which we are about to do is to examine the surface areas of buildings and to link these to energy emissions and related phenomena and when we do this, the variations in these relations with respect to different locations and districts within the city will take on new meaning. In time, we hope that such work will add to our growing knowledge of how efficient cities are in terms of their geometry and in this sense, provide a much more considered position on issues such as urban sprawl and the compact city.

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